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- USSR -

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[Following is a translation of an article by Academician A. N. Kolmogorov, Ye. F. Mishchenko, and Academician L. S. Pontryagin of the Mathematical Institute imeni V. A. Steklov, Academy of Sciences USSR, in the Russian-language periodical Doklady Akademii nauk SSSR (Reports from the Academy of Sciences USSR), Vol 145, No 5, Moscow 1962, pages 993-995.]

Let $p(\sigma, x, \tau, y)$ — be the probability density of Markov's process in an n-dimensional Euclidian space R_n $(n \geqslant 3)$, subject to Kolmogorov's equation(1)

$$\frac{\partial p}{\partial \sigma} + \sum_{i,j=1}^{n} a^{ij}(\sigma, x) \frac{\partial^2 p}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(\sigma, x) \frac{\partial p}{\partial x^i} = 0. \quad (1)$$

Let the second point z move in the same space R_n in accordance with the law $z = z(t)$. The vicinity of point z moves along with it; this space is bounded by a closed surface $\Sigma_t = z(t) + \varepsilon\Sigma$, which is similar to a fixed surface Σ, the similarity coefficient ε being small (for simplicity we shall consider Σ a sphere of a unit radius in the following text). It is required that we determine the probability $\psi(\sigma, x, \tau)$ that the random point whose transition density is subject to, (1) for example, will intersect the surface Σ_t within the time interval $< t < \tau$.

This problem was solved by Ye. F. Mishchenko and L. S. Pontryagin in their work(2) related to the needs of optimal control. However, the approximate formula for the probability ψ, obtained in this work proved to be cumbersome and ill suited for further utilization.

A. N. Kolmogorov having familiarized himself with reference (2) proposed on the basis of consideration of probabilities another considerably simpler expression for the approximation of Ye. F. Mishchenko and L. S. Pontryagin. However, he offered no proof.

In the present article there are given Kolmogorov's formula and its proof proposed by Ye. F. Mishchenko and L. S. Pontryagin. This proof is based on expressions cited in reference (2).

It is known (cf (2)) that the desired probability $\psi(\sigma, x, \tau)$ is a solution of equation (1) and satisfies the requirements

$$\psi(\tau, x, \tau) = 0, \quad \psi(\sigma, x, \tau)\,|_{\Sigma_\sigma} = 1. \tag{2}$$

A. N. Kolmogorov proposed the following formula for the principal portion $K(\sigma, x, \tau, \varepsilon)$ of the probability ψ:

$$K(\sigma, x, \tau, \varepsilon) = \varepsilon^{n-2}\int_\sigma^\tau p(\sigma, x, s, z(s))\,\beta(s)\,ds, \tag{3}$$

where

$$\beta(s) = \int_{A_s\Sigma} \frac{\partial w(s, \xi)}{\partial n}\,d\Sigma; \tag{4}$$

A_s is a linear transform in $\xi = A_s\bar{\xi}$, which reduces the differential form $\sum a^{ij}(s, z(s))\frac{\partial^2}{\partial\xi^i\partial\xi^j}$ to the form $\sum_{k=1}^{n}\frac{\partial^2}{\partial\xi^{k^2}}$, and $w(s, \xi)$ is a harmonic function satisfying the requirements

$$w(s, \xi) = 1 \quad \text{for} \quad \xi \in A_s \Sigma; \qquad w(s, \xi) \to 0 \quad \text{for} \quad |\xi| \to \infty.$$

Indirectly it is verified that the function $K(\sigma, x, \tau, \varepsilon)$, determined by formula (3) satisfies equation (1) outside of point $z(\sigma)$.

We shall show that in a certain specially selected samll ellipsoid with its center at point $z(\sigma)$, the function $K(\sigma, x, \tau, \varepsilon)$ and the function $\Psi(\sigma, x, \tau, \varepsilon)$, which is constructed in reference (2) and which is the principal portion of the probability $\psi(\sigma, x, \tau)$, do not differ "in essence", that is, they coincide with an accuracy of $O(\varepsilon)$ for $\tau - \sigma \geqslant \varepsilon$ and they differ only by $O(1)$ for $\tau - \sigma < \varepsilon$. Hence, by virtue of lemma 3 of reference (2) it follows that

$$K(\sigma, x, \tau, \varepsilon) = \Psi(\sigma, x, \tau, \varepsilon).$$

In order to prove it, we shall introduce in the space (z, t) new coordinates defined by formulas $z = \bar{\zeta} + z(t)$, $\sigma \leqslant t \leqslant s$, so that $x = \bar{\xi} + z(\sigma)$, $y = \bar{\eta} + z(s)$. Let us assume further that $\xi = A_\sigma \bar{\xi}$. For this substitution of coordinates the function $K(\sigma, x, \tau, \varepsilon)$ will be transformed into function $Q(\sigma, \xi, \tau, \varepsilon)$, and the function $\Psi(\sigma, x, \tau, \varepsilon)$ into function $\Phi(\sigma, \xi, \tau, \varepsilon)$. Apparently,

$$Q(\sigma, \xi, \tau, \varepsilon) = \varepsilon^{n-2} \int_\sigma^\tau q(\sigma, \xi, s, 0)\, \beta(s)\, ds, \tag{5}$$

where

$$q(\sigma, \xi, s, \eta) = p(\sigma, A_\sigma^{-1}\xi + z(\sigma), s, A_s^{-1}\eta + z(s)). \tag{6}$$

The function $q(\sigma, x, \tau, \eta)$ is a fundamental solution of the parabolic equation obtained from equation (1) with a substitution of coordinates ξ.

In reference (2) it is shown that the function $\Phi(\sigma, \xi, \tau, \varepsilon)$ for $|\xi| = \varepsilon$ differs only "non-essentially" from the magnitude $\alpha(\sigma)$, which is obtained in the following manner. Let us solve the Dirichlet problem for the equation $\Delta w = 0$ subject to conditions

$$w(\sigma, \xi)|_{H_\varepsilon} = 1, \qquad w(\sigma, \xi) \to 0 \quad \text{for} \quad |\xi| \to \infty.$$

Here H_ε is an ellipsoid obtained from the sphere $\varepsilon\Sigma$ by transformation A_σ. As we know, the function $w(\sigma, \xi)$ may be presented in the form

$$w(\sigma, \xi) = \frac{\varepsilon^{n-2}\alpha(\sigma)}{r^{n-2}(\xi)} + \Pi(\sigma, \xi, \varepsilon), \tag{7}$$

where $\Pi(\sigma, \xi, \varepsilon)$ is a potential of a double layer formed by the ellipsoid H_ε at point ξ.

It is not difficult to establish the relationship between $\alpha(\sigma)$ and $\beta(\sigma)$, which figure in formula (4). Indeed, if we take into account the fact that the integral of the normal derivative on surface H_ε with respect to potential of the double layer $\Pi(\sigma, \xi, \varepsilon)$ is equal to zero, then differentiating the right and left portions of expressions (7) along the normal to H_ε and then integrating over H_ε, we will demonstrate that

$$\beta(\sigma) = \frac{4\pi^{n/2}}{\Gamma(n/2 - 1)}\alpha(\sigma), \tag{8}$$

where Γ is a gamma-function.

We shall show that Kolmogorov's function $Q(\sigma, \xi, \tau, \varepsilon)$, determined by formula (5) also differs only "non-essentially" from $\alpha(\sigma)$ for $|\xi| = \varepsilon$.

Utilising the considerations and evaluations of reference(2) it can be shown first of all that

$$Q(\sigma, \xi, \tau, \varepsilon)\Big|_{|\xi|=\varepsilon} = \varepsilon^{n-2}\int_{\sigma}^{\tau}\gamma(\sigma, \xi, s, 0)\Big|_{|\xi|=\varepsilon}\beta(s)\,ds + o(1), \quad (9)$$

where $\gamma(\sigma, \xi, s, \eta)$ is Green's function of the heat transmission equation:

$$\gamma(\sigma, \xi, s, \eta) = \frac{1}{[4\pi(s-\sigma)]^{n/2}}e^{-|\xi-\eta|^2/4(s-\sigma)}. \quad (10)$$

Let us calculate the magnitude of the $\varepsilon^{n-2}\int_{\sigma}^{\tau}\gamma(\sigma, \xi, s, 0)\beta(s)\,ds$ for $|\xi| = \varepsilon$. We have

$$\varepsilon^{n-2}\int_{\sigma}^{\tau}\gamma(\sigma, \xi, s, 0)\Big|_{|\xi|=\varepsilon}\beta(s)\,ds =$$

$$= \varepsilon^{n-2}\int_{\sigma}^{\tau}\gamma(\sigma, \xi, s, 0)\Big|_{|\xi|=\varepsilon}[\beta(\sigma)\beta(s) - \beta(\sigma)]\,ds =$$

$$= \frac{\beta(\sigma)\varepsilon^{n-2}}{(4\pi)^{n/2}}\int_{\sigma}^{\tau}\frac{1}{(s-\sigma)^{n/2}}e^{-\varepsilon^2/4(s-\sigma)}ds +$$

$$+ \varepsilon^{n-2}\int_{\sigma}^{\tau}\gamma(\sigma, \xi, s, 0)\Big|_{|\xi|=\varepsilon}[\beta(s) - \beta(\sigma)]\,ds. \quad (11)$$

Let $s - \sigma = \varepsilon^2 t$. Then

$$\frac{\beta(\sigma)}{(4\pi)^{n/2}}\int_{\sigma}^{\tau}\frac{1}{(s-\sigma)^{n/2}}e^{-\varepsilon^2/4(s-\sigma)}ds = \frac{\beta(\sigma)}{(4\pi)^{n/2}}\int_{0}^{\infty}\frac{1}{t^{n/2}}e^{-1/4t}\,dt + \omega(\varepsilon, \sigma, \tau),$$

where $\omega(\varepsilon, \sigma, \tau)$ is limited for $\tau - \sigma \leqslant \varepsilon$ and we have a magnitude of the order of $O(1)$ for $\tau - \sigma > \varepsilon$. Carrying out the substitution

$x = 1/4t$, we shall obtain

$$\frac{\beta(\sigma)}{(4\pi)^{n/2}} \int_0^\infty \frac{1}{t^{n/2}} e^{-1/4t} dt = \frac{\beta(\sigma)}{4\pi^{n/2}} \Gamma\left(\frac{n}{2} - 1\right) = \alpha(\sigma). \tag{12}$$

Further, it can be easily demonstrated that

$$\varepsilon^{n-2} \int_\varepsilon^\tau \gamma(\sigma, \xi, s, 0)\big|_{|\xi|=\varepsilon} [\beta(s) - \beta(\sigma)]\, ds = o(1). \tag{13}$$

BIBLIOGRAPHY OF CITED WORKS

1. A. N. Kolmogoroff, Math. Ann., 104, 415 (1931).

2. Ye. F. Mishchenko, L. S. Pontryagin. IZV. AN USSR, Mathem. Ser. matem., 25, 477, 1961. (Proc. Acad. Sci. USSR, Mathem. Ser. 25, 477 1961.

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